On the homeotopy group of the non-orientable surface of genus three

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Abstract. In this note we prove that, if \( N_3 = P#P#P \), where \( P := \mathbb{R}P^2 \), then the canonical homomorphism from \( \text{Diff}(N_3) \) onto the homeotopy group \( \text{Mod}(N_3) \) has a section. To do this we first prove that \( \text{Mod}(N_3) = GL(2, \mathbb{Z}) \).

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Resumen. En esta nota probamos que, si \( N_3 = P#P#P \), donde \( P := \mathbb{R}P^2 \), entonces el homomorfismo canónico de \( \text{Diff}(N_3) \) sobre el grupo de homeotopía \( \text{Mod}(N_3) \) tiene una sección. Para hacer esto, primero probamos que \( \text{Mod}(N_3) = GL(2, \mathbb{Z}) \).

1. Introduction

If \( M \) is a closed smooth surface we denote by \( \text{Mod}(M) \) the quotient group \( \text{Diff}(M)/\text{Diff}_0(M) \) where \( \text{Diff}(M) \) is the group of all diffeomorphisms from \( M \) to \( M \) and \( \text{Diff}_0(M) \) is the normal subgroup of diffeomorphisms isotopic to the identity. We call it the homeotopy group or the extended mapping class group of \( M \).

S. Morita [9], [10] has shown that, if \( M_g \) is the closed genus \( g \) orientable surface, then the canonical epimorphism

\[ \text{Diff}(M_g) \to \text{Mod}(M_g) \]

from the group of diffeomorphisms of \( M_g \) onto its extended mapping class group admits no section provided that \( g \geq 18 \).

When \( g \leq 1 \) it is easy to show that the homomorphism does have a splitting: If \( g = 0 \) then \( \text{Mod}(M_0) = \mathbb{Z}_2 \); a section is defined by sending the non-trivial element of \( \text{Mod}(M_0) \) to the antipodal map of \( S^2 \). Also, for genus one \( M_1 = \)
$\mathbb{R}^2/\mathbb{Z}^2$ and $\text{Mod}(M_1) = GL(2, \mathbb{Z})$ (cf. [11, p. 26]). The standard linear action of $GL(2, \mathbb{Z})$ on $(\mathbb{R}^2, \mathbb{Z}^2)$ defines a splitting of $\text{Diff}(M_1) \to \text{Mod}(M_1)$.

If $N_k$ is the genus $k$ non-orientable surface (the connected sum of $k$ copies of $P$) then

$$\text{Diff}(N_k) \to \text{Mod}(N_k),$$

has a section if $k \leq 2$.

For, if $k = 1$ then $\text{Mod}(P) = 1$ (see [4]) and trivially a section exists. If $k = 2$ and we think of $N_2$ as $S^1 \times S^1$ with identifications $(z, w) \sim (z, -w)$, then $\text{Mod}(N_2) = \mathbb{Z}_2 \oplus \mathbb{Z}_2$ and the image of a section is $\{ f_{\varepsilon_1, \varepsilon_2} : |\varepsilon_1| = |\varepsilon_2| = 1 \}$ where

$$f_{\varepsilon_1, \varepsilon_2}(z, w) = (z^{\varepsilon_1}, w^{\varepsilon_2})$$

(see [6], [12]).

Here we will prove that

$$\text{Diff}(N_k) \to \text{Mod}(N_k),$$

also has a section if $k = 3$.

2. Proofs

First, we will show that $\text{Mod}(N_3) = GL(2, \mathbb{Z})$.

In [2], using [3], a presentation of $\text{Mod}(N_3)$ is given and one can see that this presentation defines $GL(2, \mathbb{Z})$, (see [7]).

However we feel that this result is not well known. In here we will give a proof of the fact that $\text{Mod}(N_3) = \text{Mod}(M_1)(= GL(2, \mathbb{Z})$) using simple methods in algebraic topology.

We will work in the smooth category.

Let $T_0$ be a torus minus the interior of a 2–disk $D$. An arc $\alpha$ properly embedded in $T_0$ is trivial if there is a 2–disk in $T_0$ whose boundary is the union of $\alpha$ and an arc in $\partial T_0$. This is equivalent to the condition that $\alpha$ represent the trivial element of $H_1(T_0, \partial T_0; \mathbb{Z}_2)$. In the following lemma $\cup_{i=1}^n \alpha_i/\varphi$ will denote the quotient space of the union of arcs $\cup_{i=1}^n \alpha_i$ obtained by identifying $x \in \partial(\cup_{i=1}^n \alpha_i)$ with $\varphi(x)$.

**Lemma 2.1.** Let $T_0$ be the torus minus the interior of a 2–disk. Let $\varphi: \partial T_0 \to \partial T_0$ be a fixed point free involution. Let $\alpha_1, \ldots , \alpha_n$, with $n$ odd, be disjoint arcs properly embedded in $T_0$ such that $\varphi(\cup_{i=1}^n \alpha_i) = \partial(\cup_{i=1}^m \alpha_i), \cup_{i=1}^m \alpha_i/\varphi$ is connected and $\sum_{i=1}^n |\alpha_i| = 0 \in H_1(T_0, \partial T_0; \mathbb{Z}_2) = \mathbb{Z}_2 \oplus \mathbb{Z}_2$. Then at least one $\alpha_i$ is trivial.

**Proof.** Let $a, b, c$ be the nontrivial elements of $H_1(T_0, \partial T_0; \mathbb{Z}_2)$. Let $a_1, \ldots , a_p$ be the arcs of $\{ \alpha_1, \ldots , \alpha_n \}$ which represent $a$. Let $b_1, \ldots , b_q$ those which represent $b$ and $c_1, \ldots , c_r$ those which represent $c$.

Assume no $\alpha_i$ is trivial, that is $|\alpha_i| \neq 0$ for all $i$. Then $0 = \sum_{i=1}^n |\alpha_i| = pa + bq + rc$ in $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ and $p + q + r = n$, an odd number. If one of the numbers $p, q, r$ is even then the other two must also be even: but this contradicts the fact that $n$ is odd. Therefore $p, q, r$ are all odd.

Notice that for any $i$ and any $j$ the 0–spheres $\partial \alpha_i$ and $\partial b_j$ are linked in $\partial T_0$ (meaning that both components of $\partial T_0 - \partial \alpha_i$ contain one point of $\partial b_j$).
Similarly ∂b_i and ∂c_k are linked, and ∂c_k and ∂a_j are linked, for any values of i, j, k. Also ∂a_i and ∂a_j are not linked ∂b_i and ∂b_j are not linked, ∂c_i and ∂c_k are not linked if i ≠ j.

This implies that after renumbering the a’s, b’s and c’s the arrangement of the points of \( \bigcup_{i=1}^{n} \partial a_i \) in \( \partial T_0 \) is: \( a_1^+, a_2^+, \ldots, a_p^+, b_1^+, b_2^+, \ldots, b_q^+, c_1^+, c_2^+, \ldots, c_r^+ \), as shown in figure 1; here \( \partial a_i = \{ a_i^+, a_i^- \} \), \( \partial b_j = \{ b_j^+, b_j^- \} \) and \( \partial c_k = \{ c_k^+, c_k^- \} \).

But then the number of components of \( \bigcup a_i / \phi \) is \( p + 1 + q + 1 + r + 1 > 1 \) (think of \( \phi \) as the antipodal involution), contradicting that \( \bigcup a_i / \phi \) is connected. Hence at least one \( a_i \) is trivial. □✓

We write \( N = N_3 \) henceforth.

**Proposition 2.1.** Let \( \mu \) and \( \alpha \) be simple closed curves in \( N \) representing the element of order 2 in \( H_1(N; \mathbb{Z}) \). Then \( \alpha \) is isotopic to \( \mu \).

**Proof.** Write \( N = T_0 \cup P_0 \), the union of a punctured torus \( T_0 \) and a Möbius band \( P_0 \), with \( T_0 \cap P_0 = \partial T_0 = \partial P_0 \). We think of \( P_0 \) as an I–bundle over the circle and denote by \( \phi : \partial T_0 \to \partial T_0 \) the fixedpoint free involution that interchanges the boundary points of each fiber.

We may assume that \( \mu \) is the image of a section of this bundle. We may also assume that \( \alpha \) intersects \( \partial T_0 \) minimally, that is, \( |\alpha' \cap \partial T_0| \geq |\alpha \cap \partial T_0| \) for any curve \( \alpha' \) ambient isotopic to \( \alpha \). We claim that \( |\alpha \cap \partial T_0| = 0 \).

Suppose \( |\alpha \cap \partial T_0| > 0 \). Then we can assume that \( \alpha \cap P_0 \) consists of \( n \) I–fibers \( f_1, \ldots, f_n \) and \( \alpha \cap T_0 \) is the union of \( n \) disjoint arcs \( \alpha_1, \ldots, \alpha_n \) properly embedded in \( T_0 \). As \( H_1(N) = H_1(P_0, \partial P_0) \oplus H_1(T_0, \partial T_0) = \mathbb{Z}_2 \oplus \mathbb{Z}_2 \) and as \( \alpha \) represents the element of order two, then we must have that \( \sum \left[ f_i \right] \neq 0 \) in
\[ H_1(P_0, \partial P_0; \mathbb{Z}_2) \] (that is, \( n \) must be odd) and \( \sum [\alpha_i] = 0 \) in \( H_1(T_0, \partial T_0; \mathbb{Z}_2) \). By Lemma 2.1, at least one \( \alpha_i \) must be trivial and so we can isotope \( \alpha_i \) to reduce the number of components of its intersection with \( \partial T_0 \). This contradicts our minimality assumption. Hence \( |\alpha \cap \partial T_0| = 0 \) and, since \( \alpha \) is not trivial, it is isotopic to \( \mu \).

**Proposition 2.2.** Let \( N = T_0 \cup P_0 \) with \( T_0 \cap P_0 = \partial T_0 = \partial P_0 \). Then any diffeomorphism \( h \) of \( N \) is isotopic to one leaving \( T_0 \) and \( P_0 \) invariant.

**Proof.** Let \( \mu \) be the image of a section of \( P_0 \). By Proposition 2.1, \( h\mu \) is ambient isotopic to \( \mu \) so we may assume that \( h \) leaves \( \mu \) invariant. But then we can also assume that it leaves its tubular neighborhood \( P_0 \) invariant.

**Theorem 2.3.** The natural homomorphism
\[ \psi: \text{Mod}(N) \to \text{Aut}(H_1(N)/\text{Torsion}(H_1(N)) (\cong GL(2, \mathbb{Z})) , \]
is an isomorphism.

**Proof.** Again write \( N = T_0 \cup P_0 \) and \( T = T_0 \cup D \). Any automorphism of \( H_1(T) \) is induced by a diffeomorphism of \( T \) which can be isotope so that the 2–disk \( D \) is invariant. Hence any automorphism of \( H_1(T_0, \partial T_0) \) is induced by a diffeomorphism of \( T_0 \). Since any diffeomorphism of \( \partial P_0 \) can be extended to a diffeomorphism of \( P_0 \) (a nice exercise), it follows that any automorphism of \( H_1(N)/\text{Torsion}(H_1(N)) \) is induced by a diffeomorphism of \( N \). Thus \( \psi \) is an epimorphism.

Suppose now that \( \psi(h) \) is the identity. By Proposition 2.2, \( h \) is isotopic to a diffeomorphism leaving \( T_0 \) invariant. Now, \( h|_{T_0} \) induces the identity on \( H_1(T_0, \partial T_0) \) and is therefore isotopic to \( id_{T_0} \) and a diffeomorphism of \( P_0 \) which is the identity on \( \partial P_0 \) is isotopic rel \( \partial \) to \( id_{P_0} \). Hence \( h \) is isotopic to \( id_N \). This proves that \( \psi \) is a monomorphism.

**Theorem 2.4.** The natural homomorphism \( \text{Diff}(N) \to \text{Mod}(N) \) has a section.

**Proof.** Let \( T = \mathbb{R}^2/\mathbb{Z}^2 \). Consider the blow up \( B(T) \) of \( T \) at the identity element \( e \) of \( T \). Recall \( B(T) = (T - \{e\}) \cup P^1 \) where \( P^1 \) is the space of one-dimensional vector subspaces of \( \mathbb{R}^2 \). The blow up \( B(T) \) is diffeomorphic to \( N \).

If \( f \) is a linear automorphism of \( \mathbb{R}^2 \) with \( f(\mathbb{Z}^2) = 2\mathbb{Z}^2 \), it induces a diffeomorphism of \( T - \{e\} \), a diffeomorphism of \( P^1 \) and a diffeomorphism of \( B(T) \) (cf.[3, Lemma 2.1]). Thus the standard linear action of \( GL(2, \mathbb{Z}) \) on \( T \) induces an action of \( GL(2, \mathbb{Z}) \) on \( B(T) \).

Hence we have a homomorphism
\[ GL(2, \mathbb{Z}) \to \text{Diff}(B(T)), \]
which composed with
\[ \text{Diff}(B(T)) \to \text{Mod}B(T) \xrightarrow{\cong} \text{Aut}(H_1(N)/\text{Torsion}(H_1(N)) \]
is an isomorphism.
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References


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